Monte Carlo Methods for American Options

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European vs. American/Bermudan Options

- Contrary to European options, American and Bermudan options can be exercised on multiple days up to trade expiry.
- Bermudan options can be exercised on a discrete set of dates while American options can be exercised in continuous time intervals.
- More choices, more value:
  \[
  \text{European} \leq \text{Bermudan} \leq \text{American}
  \]
- In the following we will mainly restrict our discussion to the case of Bermudan options as it is the one of more relevance in practice.
- The case of American options can be obtained as a limiting case when the number of exercise dates per unit time tends to infinity.
Exercise Policies and Stopping Times

- We indicate with \( T_1, \ldots, T_B \), the exercise dates of the option and with \( D(t) \) the deterministic set of exercise dates \( T_i \) larger or equal to time \( t \), namely \( D(t) = \{ T_i \geq t \} \).

- An exercise policy is represented mathematically by a stopping time taking values in \( D(t) \). Recall that a random variable \( \tau \) is a stopping time if the event \( \{ \tau \leq t \} \) can be determined using only the information available up to time \( t \).

- We indicate with \( \mathcal{T}(t) \) the set of stopping times taking values in \( D(t) \).

- We assume \( t < T_{B-1} \) as in the last period the Bermudan option becomes European.
Optimal Exercise

- A rational investor will exercise the option that she holds in such a way to maximize its economic value.
- As a result, the value of a Bermudan option is the supremum of the option value over all the possible exercise policies, namely

\[
\frac{V(t)}{N(t)} = \sup_{\tau \in T(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{N(\tau)} \right]
\]

where \( E(t) \) is the exercise value of the option, and \( N(t) \) is the chosen numeraire.

- In this equation \( V(t) \) is the value of the option with early exercise conditional on exercise not taking place strictly before time \( t \).
Continuation or Hold Value

- Indicate with $\eta(t)$ the smallest integer such that $T_{\eta(t)+1} > t$.
- The **hold value** $H_{\eta}(t)$ ($T_{\eta}(t) \leq t < T_{\eta}(t)+1$) is the value of the Bermudan option when the exercise dates are restricted to $D(T_{\eta(t)+1})$

\[
\frac{H_{\eta}(t)}{N(t)} = \mathbb{E}_{t} \left[ \frac{V(T_{\eta+1})}{N(T_{\eta+1})} \right]
\]

- Clearly $H_{\eta}(t) = V(t)$ for $T_{\eta} < t < T_{\eta+1}$, since there are no exercise opportunities in this interval.
- Instead $H_{\eta}(T_{\eta})$ can be interpreted as the hold value of the Bermudan option at time $T_{\eta}$, i.e., the value of the option if we decide *not* to exercise at time $T_{\eta}$.
Optimal Exercise and Dynamic Programming

▶ The option holder following an optimal exercise policy will exercise her option if the exercise value is larger than the hold value

\[ V(T_\eta) = \max(E(T_\eta), H_\eta(T_\eta)) \]

▶ This, when combined with the definition of hold value, leads to the so-called *dynamic programming* or Bellman principle formulation, namely,

\[ \frac{H_\eta(t)}{N(t)} = \mathbb{E}_t \left[ \max \left( \frac{E(T_{\eta+1})}{N(T_{\eta+1})}, \frac{H_{\eta+1}(T_{\eta+1})}{N(T_{\eta+1})} \right) \right] \]

for \( T_\eta \leq t < T_{\eta+1}, \) and \( \eta = 1, \ldots, B - 1. \)
Optimal Exercise Time

- Starting from the terminal condition $H_B(T_B) \equiv 0$, this defines a backward iteration in time for $H_\eta(T_\eta)$.
- By definition, this is also equal to $V(t)$ if $t$ is not an exercise date.
- Conversely, if $t$ is an exercise date, $t = T_\eta$, then

$$V(T_\eta) = \max(E(T_\eta), H_\eta(T_\eta))$$

- The dynamic programming formulation above implies that the stopping time defining optional exercise (as seen as time $t$) is given by

$$\tau^* = \inf[T_i \geq t : E(T_i) \geq H_i(T_i)]$$
Example: American put option

Consider an American put option struck at $K$ on a stock $S(t)$:

$$
\frac{dS(t)}{S(t)} = r dt + \sigma dW_t
$$

where $r$ is the (constant) instantaneous risk free rate of interest, $\sigma$ is the volatility and $W_t$ is a standard Brownian motion.

The value of the Bermudan put option can be expressed as

$$
V(t) = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_t \left[ e^{-r(\tau-t)} (K - S(\tau))^+ \right]
$$
Example: American put option (cont’d)

The optimal exercise time is given by

\[ \tau^* = \inf\{t \in \mathcal{D}(t) : (K - S(t))^+ \geq H(t)\} \]

Since \( H(t) \) is a function of \( S(t) \), the latter condition can be expressed equivalently as

\[ \tau^* = \inf\{t \in \mathcal{D}(t) : S(t) < b^*(t)\} \]

for a deterministic function \( b^*(t) \), assuming the natural interpretation of *exercise boundary*. 
Example: American put option (cont’d)

Figure: Exercise Boundary for the American Put option on a single stock (taken from Ref. [1]).
Monte Carlo challenges

- The dynamic programming recursion can be easily implemented in the context of deterministic numerical methods (multinomial trees or PDEs), which are based on *backward induction*.
- These are limited by the *curse of dimensionality*.
- On the other hand, in the context of Monte Carlo methods, the paths describing the time evolution of the underlying risk factors are generated *forward in time*, thus making the direct application of backward induction impossible.
- This makes pricing Bermudan options, whose dimensionality is too high to be treated by deterministic numerical methods, very challenging.
Variational Principle and Lower Bound Methods

- An immediate consequence of

\[
\frac{V(t)}{N(t)} = \sup_{\tau \in T(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{N(\tau)} \right]
\]

is that for any stopping time \( \tau \in T(t) \)

\[
\frac{V(t)}{N(t)} \geq \mathbb{E}_t \left[ \frac{E(\tau)}{N(\tau)} \right]
\]

- As a result, a lower bound for the value of the Bermudan option \( V(t) \)
can be computed by means of Monte Carlo through any exogenous guess for the optimal exercise strategy \( \tau^* \).
Parametric Exercise Boundary Methods

- Parametric lower bound methods involve a user supplied specification of a parametric stopping rule $\tau_\theta \in \mathcal{T}_\theta(t)$ where $\theta \in \Theta \subset \mathbb{R}^{N_\theta}$ is a $N_\theta$ dimensional parameter vector.

- Due to the variational principle, for any value of $\theta$ the stopping rule generates a lower bound, $V_\theta(t)$, of the true value of the Bermudan option, $V(t)$.

- Clearly, the best approximation of such value within the chosen class, is the one for which the value $V_\theta(t)$ is the largest, namely

$$V_{\theta^*}(t) = \sup_{\theta} V_\theta(t)$$
Algorithm

Step 1 Generate $n$ independent paths of the underlying Markov process $X^{(k)} = (X^{(k)}(T_{\eta(t)}+1), \ldots, X^{(k)}(T_B))$, for $k = 1, \ldots, n$. For path $k$, let $\tau^{(k)}(\theta)$ be the exercise time suggested by the stopping rule for the given value of the parameter $\theta$.

Step 2 For each path $k$, set:

$$E_{\theta}^{(k)} = E(X(\tau_{\theta}^{(k)}))$$

$$V_{\theta}^{(k)} = \frac{E_{\theta}^{(k)}}{N(X(\tau_{\theta}^{(k)}))}$$
Algorithm (cont’d)

Step 3 Return:

\[ \tilde{V}_\theta(t) = N(t) \times \frac{1}{n} \sum_{k=1}^{n} V^{(k)}_{\theta} \]

Step 4 Find:

\[ \theta^* = \arg \sup_{\theta \in \Theta} \tilde{V}_\theta(t) \]

Step 5 Return:

\[ \tilde{V}_{\theta^*}(t) = \sup_{\theta \in \Theta} \tilde{V}_\theta(t) \]
The value \( \tilde{V}_{\theta^*}(t) \) is an estimate of the true value of the Bermudan option \( V(t) \). However we do not know whether it is a lower bound or an upper bound.

This is because while \( V_{\theta^*}(t) \leq V(t) \), because of the variational principle, the estimator \( \tilde{V}_{\theta^*}(t) \) is instead biased high with respect to \( V_{\theta^*}(t) \).

Indeed, as a result of Jensen’s inequality:

\[
\mathbb{E}[\tilde{V}_{\theta^*}(t)] = \mathbb{E}[\sup_{\theta \in \Theta} \tilde{V}_\theta(t)] \geq \sup_{\theta \in \Theta} \mathbb{E}[\tilde{V}_\theta(t)] = \sup_{\theta \in \Theta} V_\theta(t) = V_{\theta^*}(t)
\]

so that:

\[
\mathbb{E}[\tilde{V}_{\theta^*}(t)] \geq V_{\theta^*}(t) \leq V(t)
\]
Algorithm: Lower Bound!

- In order to avoid this problem on can replace Step 5 with the following two steps:

  **Step 5’** Draw $N$ independent paths for $X^{(k)}$ and compute for each path:

  $$E_{\theta^*}^{(k)} = E(X^{(k)}(\tau_{\theta^*}))$$

  $$V_{\theta^*}^{(k)} = \frac{E_{\theta^*}^{(k)}}{N(X^{(k)}(\tau_{\theta})))}$$

- **Step 6** Return:

  $$\bar{V}_{\theta^*}(t) = N(t) \times \frac{1}{N} \sum_{k=1}^{N} V_{\theta^*}^{(k)}$$

- Since $E[\bar{V}_{\theta^*}(t)] = V_{\theta^*}(t) \leq V(t)$, this provides a genuine lower bound estimator.
The optimization in Step 4 above can be simplified whenever the parameter $\theta$ decomposes into $B - 1$ sub-components, with the $i$-th subset parametrizing the exercise decision at time $T_i$, i.e. $\theta = (\theta_{\eta(t)+1}, \ldots, \theta_{B-1})$ with each $\theta_i$ possibly a vector itself.

For instance, this is the case for the so-called *moneyness stopping rule*

$$\tau(\theta) = \inf_{i > \eta(t)} [T_i : E(X(T_i)) > \theta_i]$$

prescribing early exercise whenever the option is in the money ‘deeply enough’.
A Useful Parameterization of the Exercise Boundary (cont’d)

- This can be seen as a simplification of the optimal stopping rule

\[ \tau^* = \inf\left[ T_i \geq t : E(T_i) \geq H_i(T_i) \right] \]

in which the hold value, \( H_i(X(T_i)) \), is replaced by the constant \( \theta_i \).

- In this case, Step 4 above is replaced by the following step:

**Step 4’** Proceeding backwards in time, for \( i = B - 1, \ldots, \eta(t) + 1 \) find \( \tilde{\theta}_i \) by keeping \((\tilde{\theta}_{i+1}, \ldots, \tilde{\theta}_{B-1})\) fixed and maximizing:

\[ \tilde{V}(t) = N(t) \times \sum_{k=1}^{n} \frac{E(X(\tilde{\tau}_i))}{N(\tilde{\tau}_i)} \]  \hspace{1cm} (1)

where \( \tilde{\tau}_i = \tilde{\tau}(\theta_i, \tilde{\theta}_{i+1}, \ldots, \tilde{\theta}_{B-1}) \in D(T_i) \).
A Useful Parameterization of the Exercise Boundary (cont’d)

Note that:

- Step 4’ does not involve repeating the Monte Carlo simulation from scratch. Rather the same set of $n$ paths can be reused.
- With a finite number of paths the one in Step 4’ is a non-smooth optimization problem and is best solved by an iterative search rather than a derivatives based approach.
- There is no guarantee the algorithm in Step 4’ produces the optimum $\theta^* = \arg \sup_{\theta \in \Theta} \tilde{V}_{\theta}(t)$. This is because each $\tilde{\theta}_i$ is optimized assuming exercise only after $T_i$. In reality, only a subset of the paths would arrive to $T_i$ without triggering early exercise beforehand.
Exercise 1

Consider a Bermudan option on the maximum of two assets following a geometric Brownian motion of the form

\[ \frac{dS_i(t)}{S_i(t)} = (r - \delta)dt + \sigma dW_i^t \]

where \( r = 5\% \), \( \delta = 10\% \), \( \sigma = 20\% \) and \( S_1(0) = S_2(0) \). The two assets are assumed independent. The (undiscounted) payoff upon exercise at time \( T_i \) is

\[ (\max(S_1(t), S_2(t)) - K)^+ \]

where \( K = 100 \) is the strike price. The maturity of the option is \( T_B = 3 \) and can be exercised at nine equally spaced dates \( T_i = i/3 \) with \( i = 1, \ldots, 9 \). The exact option prices obtained are 13.90, 8.08 and 21.34 for \( S_i(0) = 100, 110 \) and 90, respectively. Use the parametric lower bound method and the moneyness stopping rule to estimate the value of the option (use both a single value of \( \theta \) and a different value of \( \theta \) for each exercise date). Compare the results obtained with Steps 1 to 5 and Steps 1 to 6 (through Step 5\(^{\prime}\)) of the algorithm above.
Regression Approaches

- The optimal exercise strategy defined by

\[ \tau^* = \inf \left[ T_i \geq t : E(T_i) \geq H_i(T_i) \right] \]

can be approximated by constructing an estimate of the hold value

\[ H_i(T_i), \ i = \eta(t) + 1, \ldots, B - 1. \]

- In general, in a Markov setting the hold value is a function of the state vector at time \( T_i \)

\[ H_i(x) = N(x) \times \mathbb{E} \left[ \frac{V(X(T_{i+1}))}{N(X(T_{i+1}))} \middle| X(T_i) = x \right] \]
Regression Approaches (cont’d)

- Regression based approaches are based on the following *ansatz* for the hold value

\[
\hat{H}_i(x) = \sum_{j=1}^{d} \beta_{ij} \psi_j(x)
\]

for a set of *d basis functions* \(\psi_j(x)\) and coefficients \(\beta_{ij}\). Equivalently, this can be written in matrix form as

\[
\hat{H}_i(x) = \beta_i^T \psi(x) = \psi^T(x) \beta_i
\]

where \(\beta_i = (\beta_1, \ldots, \beta_d)^T\) and \(\psi(x) = (\psi_1(x), \ldots, \psi_d(x))^T\).
Regression Approaches (cont’d)

- Multiplying by $\psi^T(X(T_i))$, using the definition of hold value and taking unconditional expectations one gets

$$
\mathbb{E} \left[ \beta_i^T \psi(X(T_i)) \psi^T(X(T_i)) \right] = \mathbb{E} \left[ \frac{N(X(T_i)) V(X(T_{i+1}))}{N(X(T_{i+1}))} \psi^T(X(T_i)) \right]
$$

which gives

$$
\beta_i = \Psi_i^{-1} \Omega_i
$$

where we define the $d \times d$ matrix

$$
\Psi_i = \mathbb{E} \left[ \psi(X(T_i)) \psi^T(X(T_i)) \right]
$$

and the $d \times 1$ vector

$$
\Omega_i = \mathbb{E} \left[ \frac{N(X(T_i)) V(X(T_{i+1}))}{N(X(T_{i+1}))} \psi(X(T_i)) \right]
$$
Regression Approaches (cont’d)

▶ These equations provide a straightforward recipe to compute the regression coefficients $\beta_i$ by substituting $\Psi$ and $\Omega$ with their sample average over $n$ Monte Carlo replications, $\bar{\Psi}_i$ and $\bar{\Omega}_i$.

▶ More explicitly, considering a set of Monte Carlo paths of the Markov state variable $X$ sampled on the Bermudan exercise dates

$$(X^{(k)}(T_{\eta(t)+1}), \ldots, X^{(k)}(T_{B-1}))$$

for $k = 1, \ldots, n$ one could compute the sample averages

$$\bar{\Psi}_i = \frac{1}{n} \sum_{k=1}^{n} \psi(X^{(k)}(T_i)) \psi^T(X^{(k)}(T_i))$$

$$\bar{\Omega}_i = \frac{1}{n} \sum_{k=1}^{n} \frac{N(X^{(k)}(T_i)) \hat{V}(X^{(k)}(T_{i+1}))}{N(X^{(k)}(T_{i+1}))} \psi(X^{(k)}(T_i))$$
Variational Principle and Lower Bound Methods

Regression Approaches

Regression Approaches (cont’d)

In the last equation $\hat{V}$ is given by

$$\hat{V}(X^{(k)}(T_i)) = \max (E(X^{(k)}(T_i), \hat{H}_i(X^{(k)}(T_i))))$$

for $i = \eta(t) + 1, \ldots B - 1$, where we have replaced the true value $H$ with the estimate $\hat{H}$ according to

$$\hat{H}_i(X^{(k)}(T_i)) = \sum_{j=1}^{d} \beta_{ij} \psi_j(X^{(k)}(T_i))$$

However this depends on the yet to be determined coefficients $\beta_i$. 
Backward Induction

- As a result, the calculation of the sample averages $\bar{\Psi}_i$ and $\bar{\Omega}_i$ needs to be performed backwards. Indeed, starting from the penultimate Bermudan exercise date $T_{B-1}$ on which

$$\hat{V}(X^{(k)}(T_B)) = \max(E(X^{(k)}(T_B)), 0)$$

one can compute

$$\bar{\beta}_{B-1} = \bar{\Psi}_{B-1}^{-1} \bar{\Omega}_{B-1}$$

which allows one to compute for $i = B - 2$ the estimate of the hold value

$$\hat{H}_{B-1}(X^{(k)}(T_{B-1})) = \sum_{j=1}^{d} \bar{\beta}_{B-1}^j \psi_j(X^{(k)}(T_{B-1}))$$

required to compute the estimate $\hat{V}(X^{(k)}(T_{B-1}))$. This can be iterated until we get to $i = \eta(t) + 1$...
Algorithm

**Step 1** Simulate $n$ independent paths ($k = 1, \ldots, n$)

$$(X^{(k)}(T_{\eta(t)+1}), \ldots, X^{(k)}(T_{B-1}))$$

**Step 2** At final expiry compute the value:

$$\hat{V}(X^{(k)}(T_B)) = \max(E(X^{(k)}(T_B)), 0)$$

**Step 3** For $i = B - 1, \ldots, \eta(t) + 1$

a) Compute:

$$\bar{\Psi}_i = \frac{1}{n} \sum_{k=1}^{n} \psi(X^{(k)}(T_i))\psi^T(X^{(k)}(T_i))$$

b) Compute:

$$\bar{\Omega}_i = \frac{1}{n} \sum_{k=1}^{n} \frac{N(X^{(k)}(T_i))\hat{V}(X^{(k)}(T_{i+1}))}{N(X^{(k)}(T_{i+1}))} \psi(X^{(k)}(T_i))$$

using the value of $\hat{V}(X^{(k)}(T_{i+1}))$ computed in the previous time step.
Algorithm (cont’d)

c) Compute by matrix inversion and multiplication:
\[ \bar{\beta}_i = \bar{\Psi}_i^{-1} \bar{\Omega}_i \]

d) Set for the estimate of the hold value at time \( T_i \):
\[ \hat{H}_i(X^{(k)}(T_i)) = \bar{\beta}_i \psi_i(X^{(k)}(T_i)) \]

e) Set for the estimate of the Bermudan option value at time \( T_i \):
\[ \hat{V}(X^{(k)}(T_i)) = \max(E(X^{(k)}(T_i)), \hat{H}_i(X^{(k)}(T_i))) \]

Step 4 Compute:
\[ \bar{V}(t) = N(t) \times \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{V}(X^{(k)}(T_{\eta(t)+1}))}{N(X^{(k)}(T_{\eta(t)+1}))} \]
A modification of this algorithm was proposed by Longstaff and Schwartz and entails replacing

\[ \hat{V}(X^{(k)}(T_i)) = \max(E(X^{(k)}(T_i)), \hat{H}_i(X^{(k)}(T_i))) \]

in Step 3e, with

\[ \hat{V}(X^{(k)}(T_i)) = \begin{cases} E(X^{(k)}(T_i)) & \text{if } E(X^{(k)}(T_i)) > \hat{H}_i(X^{(k)}(T_i))) \\ \hat{V}(X^{(k)}(T_{i+1}))/N(X^{(k)}(T_i))/N(X^{(k)}(T_{i+1})) & \text{otherwise} \end{cases} \]

which in the examples considered was shown to lead to more accurate results.

Similarly to the parametric exercise boundary methods, regression based approaches produce lower bound estimate of the true Bermudan option value if this is computed by means of a second simulation in which the continuation value estimated in the first simulation is used to determine early exercise.
Exercise 2

Consider the Bermudan option of Exercise 1. Compare the results obtained with the regression based approach (including the modification of Longstaff and Schwartz). Use basis functions of the form $S_1^\alpha S_2^\beta$. Include results obtained directly from the backward induction Steps 1-4 in this Section and those obtained by means of a second independent simulation using the hold value estimated by means of the backward induction in the first simulation.
The pricing problem for American or Bermudan option admit a ‘dual’ formulation in which it can be expressed as a minimization problem.

The dynamic programming equations

\[
\frac{H_\eta(t)}{N(t)} = \mathbb{E}_t \left[ \frac{V(T_{\eta+1})}{N(T_{\eta+1})} \right]
\]

\[
V(T_\eta) = \max(E(T_\eta), H_\eta(T_\eta))
\]

imply

\[
\frac{V(X(T_i))}{N(X(T_i))} \geq \mathbb{E} \left[ \frac{V(X(T_{i+1}))}{N(X(T_{i+1}))} \mid X(T_i) \right]
\]
Super-martingale Property

- Hence the discounted value of a Bermudan option, \( V(X(T_i))/N(X(T_i)) \), is a super-martingale.

- In addition one also has:
  \[
  \frac{V(X(T_i))}{N(X(T_i))} \geq \frac{E(X(T_i))}{N(X(T_i))}
  \]

- In fact, the discounted value of a Bermudan option is the minimal super-martingale dominating the discounted exercise value \( E(X(T_i))/N(X(T_i)) \).
Duality and Upper Bounds

Consider a martingale $M$ with $M(t) = \mathbb{E}_t[M(s)] = 0$.

$$\frac{V(t)}{\mathcal{N}(t)} = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{\mathcal{N}(\tau)} \right]$$

$$= \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{\mathcal{N}(\tau)} + M(\tau) - M(\tau) \right] = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{\mathcal{N}(\tau)} - M(\tau) \right]$$

Hence, using Jensen’s inequality:

$$\frac{V(t)}{\mathcal{N}(t)} = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_t \left[ \frac{E(\tau)}{\mathcal{N}(\tau)} - M(\tau) \right] \leq \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{\mathcal{N}(\tau)} - M(\tau) \right) \right]$$

As a result an upper bound on the value of the Bermudan option is

$$\frac{V(t)}{\mathcal{N}(t)} \leq \inf_{M(t)=0} \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{\mathcal{N}(\tau)} - M(\tau) \right) \right]$$
Duality and Upper Bounds (cont’d)

- Interestingly, we can show that

\[
\frac{V(t)}{N(t)} \leq \inf_{M(t)=0} \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{N(\tau)} - M(\tau) \right) \right]
\]

holds with equality i.e., it is possible to find \( M^* \) such that

\[
\frac{V(t)}{N(t)} = \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{N(\tau)} - M^*(\tau) \right) \right]
\]
Duality and Upper Bounds (cont’d)

This is a consequence of the so called *Doob-Mayer decomposition* for super-martingales:

$$\frac{V(s)}{N(s)} = \frac{V(t)}{N(t)} + M^*(s) - A(s)$$

where $A(s)$ is a non decreasing predictable process with $A(t) = 0$. Indeed, choosing for $M$ the martingale component above

$$\frac{V(t)}{N(t)} \leq \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{N(\tau)} - M^*(\tau) \right) \right] = \frac{V(t)}{N(t)} + \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{N(\tau)} - \frac{V(\tau)}{N(\tau)} - A(\tau) \right) \right] \leq \frac{V(t)}{N(t)}$$
Duality and Upper Bounds (cont’d)

- As a result the Bermudan option value can be obtained by finding the martingale component of the deflated option price.
- Of course this is in general as difficult as the original problem. Nonetheless any approximation of such martingale component will provide an upper bound of the option price through

\[
\frac{V(t)}{N(t)} \leq \inf_{M(t) = 0} \mathbb{E}_t \left[ \max_{\tau \in \mathcal{D}(t)} \left( \frac{E(\tau)}{N(\tau)} - M(\tau) \right) \right]
\]

- Conceivably the better the approximation the tighter the upper bound.
Martingales from Approximate Option Values

- Given an approximation of the deflated option price \( \frac{V(X_s)}{N(X_s)} \) one can extract the corresponding martingale component to construct an approximation of \( M^*(s) \)
- One natural choice is

\[
\hat{M}^*(t) = \sum_{j=\eta(t)+1}^{B} \Delta(T_j)
\]

with \( M(t) = 0 \), and

\[
\Delta(T_j) = \frac{V(X(T_j))}{N(X(T_j))} - \mathbb{E} \left[ \frac{V(X(T_j))}{N(X(T_j))} \mid X(T_{j-1}) \right]
\]

for \( j > \eta(t) + 1 \) and

\[
\Delta(T_{\eta(t)+1}) = \frac{V(X(T_{\eta(t)+1}))}{N(X(T_{\eta(t)+1}))} - \mathbb{E} \left[ \frac{V(X(T_{\eta(t)+1}))}{N(X(T_{\eta(t)+1}))} \mid X(t) \right]
\]
Martingales from Approximate Option Values (cont’d)

- By applying the ‘Tower law’ of conditional expectations it is immediate to see that $\hat{M}^\star$ is indeed a martingale.
- The same is true if we replace $V$ with the estimators $\hat{V}$ defined e.g., by a regression based approach.
- However, replacing the true deflated hold value $\mathbb{E}\left[\frac{V(X(T_j))}{N(X(T_j))} \bigg| X(T_{j-1})\right]$ with an approximate one $\hat{H}/N$ would not guarantee the martingale property for $\Delta(T_j)$.
- As a result, the deflated hold value needs to be valued by means of a single time-step nested simulation of $N'$ paths spun out of $X(T_{j-1})$ namely

$$\mathbb{E}\left[\frac{V(X(T_j))}{N(X(T_j))} \bigg| X(T_{j-1})\right] \approx \frac{1}{N'} \sum_{m=1}^{N'} \frac{\hat{V}(X^{(m)}(T_j))}{N(X^{(m)}(T_j))}$$
Martingales from Approximate Stopping Times

- Alternatively one can employ approximations of the optimal stopping time.

- Denote with \( \tau_j \) the stopping time as seen at time \( T_j \), i.e. \( \tau_j \in D(T_j) \), and suppose that these stopping times are defined by approximate hold value functions \( \hat{H} \), namely,

\[
\tau_j = \min \{ k = j, \ldots, B : E(X(T_k)) \geq \hat{H}(X(T_k)) \}
\]

where \( \hat{H} \) could be estimated for instance by means of regression.

- One can construct \( \Delta(T_j) \) as

\[
\Delta(T_j) = \mathbb{E} \left[ \frac{E(X(\tau_j))}{N(X(\tau_j))} \middle| X(T_j) \right] - \mathbb{E} \left[ \frac{E(X(\tau_j))}{N(X(\tau_j))} \middle| X(T_{j-1}) \right]
\]
Also note that

\[
\mathbb{E}
\left[
\frac{E(X(\tau_j))}{N(X(\tau_j))}
\bigg| X(T_j)
\right]
\]

\[
= \begin{cases} 
  \frac{E(X(T_j))}{N(X(T_j))} & \text{if } E(X(T_j)) \geq \hat{H}(X(T_j)), \\
  \mathbb{E}
  \left[
  \frac{E(X(\tau_{j+1}))}{N(X(\tau_{j+1}))}
  \bigg| X(T_j)
  \right] & \text{otherwise.}
\end{cases}
\]

As a result the only quantities that need to be valued are those of the form

\[
\mathbb{E}
\left[
\frac{E(X(\tau_{j+1}))}{N(X(\tau_{j+1}))}
\bigg| X(T_j)
\right]
\]

which can be computed by sub-simulation.
Upper Bond Algorithm

Step 1 Generate \( n \) independent paths of the underlying Markov process \( X^{(k)} = (X^{(k)}(T_{\eta(t)+1}), \ldots, X^{(k)}(T_B)) \), for \( k = 1, \ldots, n \).

Step 2 For each path \( k \), at each \( X^{(k)}(T_j) \), \( j = \eta(t) + 1, \ldots, B \)
   a) Evaluate:

\[
E^{(k)}_j = E(X^{(k)}(T_j))
\]
\[
\hat{H}^{(k)}_j = \hat{H}(X^{(k)}(T_j))
\]
\[
\mathcal{N}^{(k)}_j = \mathcal{N}(X^{(k)}(T_j))
\]

b) Simulate \( N' \) subpaths starting from \( X^{(k)}(T_j) \) and compute

\[
\bar{E}^{(k)}_j = \frac{1}{N'} \sum_{m=1}^{N'} E(X^{(k,m)}(\tau_{j+1})).
\]

c) Compute \( \hat{\Delta}^{(k)}_j \) using the information above.
Upper Bond Algorithm

**Step 3** For each path $k$, compute: $M^{(k)}(T_i)$ for $i = \eta(t) + 1, \ldots, B$.

**Step 4** For each path $k$, evaluate: $E^{(k)}(T_i) - M^{(k)}(T_i)$ for $i = \eta(t) + 1, \ldots, B$.

**Step 5** For each path $k$, evaluate:

$$U^{(k)} = \min_{i=\eta(t)+1,\ldots,B} \left( \frac{E^{(k)}(T_i)}{N(X(T_i))} - M^{(k)}(T_i) \right).$$

**Step 6** Return:

$$\bar{V}(t) = N(t) \times \frac{1}{N} \sum_{k=1}^{N} U^{(k)}. $$
Exercise 3

Consider the Bermudan option of Exercise 1. Apply the upper bound method described in this Section. Use the different estimators of the exercise time considered in Exercise 2 in order to construct approximate martingales components of the deflated option price.
References I


